The Number Chain Under Finite Bounds

A systematic evaluation of each link when the Axiom of Infinity is removed and the Axiom of Finite Bounds is imposed

---

ℕ — Natural Numbers

Status: The chain breaks at the first link.

Under standard ZFC, ℕ exists as a completed set — the Axiom of Infinity guarantees exactly this. The Axiom of Finite Bounds directly negates that guarantee:

$$\neg\exists S[\varnothing \in S \land \forall x(x \in S \rightarrow x \cup \{x\} \in S)]$$

What survives: individual natural numbers exist. For any particular n, the finite set $\{0, 1, 2, \ldots, n\}$ exists. The successor operation $x \rightarrow x \cup \{x\}$ can be applied — but not without termination. There is some largest constructible natural number. We don't know what it is, and the paper argues we don't need to.

What doesn't survive: ℕ *as a completed object* — the set containing all natural numbers simultaneously. This is precisely what the axiom forbids. You can count very high. You cannot collect the result.

Internal coherence assessment:

Clean. This is exactly what the negation says, and it follows directly.

External pressure:

Here's the first serious difficulty the paper underplays. Without ℕ as a completed set, you lose *induction as standardly formulated*. The induction schema in Peano arithmetic quantifies over all natural numbers. If there's no completed ℕ, the status of statements like "for all n, $P(n)$" becomes delicate. The paper's modified Separation and Replacement restrict to "finitely expressible properties," but it doesn't fully specify what theory of arithmetic survives. Primitive recursive arithmetic works. But the boundary between what's provable and what isn't, in the paper's system, is left undrawn.

ℤ — Integers

Status: Finite fragment survives.

ℤ is constructed from ℕ via equivalence classes of pairs — each integer is represented as a difference $a - b$ where $a, b \in$ ℕ. If ℕ doesn't exist as a completed set, neither does ℤ.

But for any bound on the naturals, there's a corresponding finite set of integers: $\{-k, \ldots, -1, 0, 1, \ldots, k\}$ for some large k.

Concrete number theory — modular arithmetic, factoring, primality testing — operates on finite integers and is untouched. What's lost is the ability to make universal claims over all integers simultaneously as a completed domain.

$\mathbb{Q}$ — Rationals

Status: Finite fragment survives, density reinterpreted.

$\mathbb{Q}$ is constructed as pairs of integers with equivalence. Same pattern: no completed $\mathbb{Q}$, but any finite collection of rationals you need is available. Between any two rationals you can always find another — but under finite bounds, this process terminates. You can subdivide many times, not infinitely many times.

This is where the paper's framework produces a genuinely interesting reinterpretation. Classical $\mathbb{Q}$ is *dense*: between any two rationals lies another. Under finite bounds, this is reframed as: between any two rationals you've constructed, you can construct another, until you hit the bound. Density becomes a *practical capacity* rather than a completed structural property.

The honest difficulty:

Many results in number theory — Dirichlet's theorem on primes in arithmetic progressions, the prime number theorem — are stated as claims about $\mathbb{Q}$ or $\mathbb{Z}$ as infinite structures. The paper would need to show, case by case, which of these admit finite reformulations that preserve their content. It gestures at this but doesn't do the work.

$\mathbb{R}$ — The Real Continuum

Status: Does not exist. This is the critical break.

The paper correctly identifies this as the crucial link. $\mathbb{R}$ is constructed from $\mathbb{Q}$ by either Dedekind cuts (each real is defined by an infinite partition of the rationals) or Cauchy sequences (each real is an equivalence class of infinite convergent sequences). Both constructions require completed infinite sets at every step:

- A Dedekind cut is a partition of *all* of $\mathbb{Q}$ — an infinite set
- A Cauchy sequence is an infinite sequence with a convergence property
- The completeness of $\mathbb{R}$ (every bounded sequence has a limit) is an infinitary property

Under the Axiom of Finite Bounds, none of these constructions go through. There is no real number line. There is no continuum.

What replaces it:

The paper implies (though doesn't fully develop) that what exists is a very dense but discrete number system — something like the rationals with bounded denominators, or fixed-precision arithmetic with an unknown but finite precision. Computationally, this is what every machine already uses: IEEE 754 floating-point arithmetic is a finite approximation of ℝ, and it works extraordinarily well.

Internal coherence:

Solid. If you deny completed infinite sets, ℝ cannot be constructed. Full stop.

External pressure — and this is severe:

The loss of ℝ is not merely the loss of an abstract object. It's the loss of:

1. **Calculus as standardly formulated.** Limits, derivatives, integrals all presuppose the completeness of ℝ. The paper argues calculus is "a very good approximation" — and computationally this is true — but the *theoretical* framework of analysis dissolves entirely. The intermediate value theorem, the mean value theorem, the fundamental theorem of calculus: all require completeness.
2. **Measure theory.** Lebesgue measure, probability theory as formalized by Kolmogorov, the entire apparatus of modern probability — gone. These require uncountable sets and σ-algebras defined over them.
3. **The foundation for physics.** Every differential equation in physics is formulated over ℝ or $\mathbb{R}^n$. The paper acknowledges this and argues the equations work because they're approximations. But the *proofs* that solutions exist, are unique, or are stable all depend on the completeness of ℝ. Without those proofs, you have numerical evidence but no theoretical guarantees.

The paper's response — that this is all "very good approximation" — is not wrong, but it's doing a lot of heavy lifting. A finitist reconstruction of analysis that preserves the theorems practitioners actually use is a major research program, not a footnote. Weyl's *Das Kontinuum* and Bishop's constructive analysis are partial models, but even they use more infinitary reasoning than the Axiom of Finite Bounds permits.

ℂ — Complex Numbers

Status: Does not exist as a completed field.

ℂ = ℝ × ℝ with the multiplication rule $(a,b) \cdot (c,d) = (ac - bd, ad + bc)$. If ℝ doesn't exist, neither does ℂ. This is where the Millennium Problem consequences bite hardest:

> **The Riemann zeta function** is defined as $\zeta(s) = \Sigma_{n=1}^{\infty} n^{-s}$, a function on ℂ extended by analytic continuation. Without ℂ, without infinite series, without analytic continuation, the function doesn't exist, and the Riemann Hypothesis has no object to be about.

L-functions in general (Birch and Swinnerton-Dyer, Langlands program) live in ℂ and require analytic continuation.

Hodge theory uses complex manifolds and cohomology over ℂ.

The paper's verdict — "dissolves as artifact" — follows internally. If the domain doesn't exist, questions about objects in the domain don't arise.

The serious counterargument the paper should address more directly:

Finite fields $\mathbf{F}_q$ have their own zeta functions (the Weil zeta functions), and the Riemann Hypothesis *for finite fields* was proved by Deligne in 1974. The pattern that the Riemann Hypothesis detects — deep structure in the distribution of primes — manifests in finite settings. This cuts both ways: it supports the paper's claim that "finite analogs may contain the genuine content," but it also suggests that the infinite formulation was capturing something real about that structure, not creating an artifact. The infinite framework *discovered* the Weil conjectures by analogy. A purely finite mathematics might never have found them.

The Transfinite Hierarchy: $\aleph_0$, $\aleph_1$, $\aleph_2$, … and ω, ω+1, …

Status: Entirely eliminated.

This one is the most straightforward. The transfinite hierarchy is built by iterating Power Set and Replacement on infinite sets. Without the first infinite set, there's nothing to iterate on.

- $\aleph_0$ (the cardinality of ℕ) doesn't exist because ℕ doesn't exist as a completed set
- **Cantor's diagonal argument** doesn't go through because it operates on completed infinite sets
- **The continuum hypothesis** (whether there's a cardinality between $\aleph_0$ and $2^{\aleph_0}$) dissolves — neither cardinal exists
- **Large cardinal axioms** — inaccessibles, measurables, Woodin cardinals — the entire hierarchy collapses
- **Transfinite ordinals** ω, ω+1, … — none exist

Internal coherence:

Perfect. This is the cleanest consequence. If there are no infinite sets, there are no infinite cardinalities or ordinals.

The loss:

Cantor's paradise was not merely decorative. Descriptive set theory, which uses the transfinite hierarchy to classify sets of reals by complexity, has produced deep structural insights. Determinacy results (Martin's theorem, projective determinacy) connect large cardinals to the structure of definable sets in ways that feel like discoveries, not artifacts. Set theorists would argue — and not unreasonably — that the coherence and fruitfulness of the transfinite hierarchy is itself evidence that it tracks *something*, even if that something isn't physical.

The paper's counter would be: beauty and coherence don't entail existence. Ptolemaic epicycles were beautiful too. This is a genuine philosophical impasse, and the paper is honest enough to acknowledge it implicitly.

The Overall Assessment

The chain dissolves cleanly from the paper's premises. There's no logical gap in the sequence: deny the completed ℕ → lose ℤ, ℚ, ℝ, ℂ as completed objects → lose everything built on them. The internal logic is sound.

The three hard questions the paper hasn't fully answered:

First: What positive theory of arithmetic and approximation replaces the infinite chain?

The criticism is fair and worth owning directly. The paper is stronger on demolition than on construction, and that asymmetry is deliberate but not cost-free. The honest answer is that the positive theory is *not yet built*, and claiming otherwise would be dishonest.

But two things can be said. First, the paper isn't claiming to deliver a complete replacement system — it's claiming that the *audit* is warranted. The finding that precisely the controversial axioms fall away, that the crisis points in physics cluster where the infinite assumption is pushed hardest, that finite computational methods already do the heavy lifting in practice — these are reasons to *begin* the reconstruction, not a finished reconstruction.

Second, the outlines of what survives are clearer than the criticism suggests. Primitive recursive arithmetic is well-understood and sufficient for an enormous range of concrete mathematics. Bounded arithmetic (Buss's S^i_2 hierarchies), feasible arithmetic, and the mathematics that proof complexity researchers already work with provide existing frameworks for reasoning about finite structures with finite resources. The gap isn't "we have nothing"; it's "we haven't systematically mapped what we have against what practitioners actually use." That mapping is the research program. The paper should say so explicitly rather than leaving it implicit.

Second: Why does the approximation work so well?

This is the question that, if answered, would transform the paper from an interesting philosophical provocation into a genuine foundational contribution. And the shape of an answer is visible even if the details aren't worked out.

The core observation is convergence of finite methods to continuous predictions as precision increases. This isn't mysterious — it's the content of numerical analysis as an entire field. When you approximate an integral with a Riemann sum, the error bound is a function of step size, and it shrinks predictably. When you solve a differential equation with finite differences, convergence theorems tell you how quickly the discrete solution approaches the continuous one. The relevant point is that these convergence results can themselves be stated and proved in finite terms — they're assertions about finite sums, finite differences, and computable error bounds. You don't need the completed real line to prove that a finite approximation of a derivative is within ε of the value that the continuous formalism predicts, for any concrete ε you specify.

What the paper needs is something like: "For any theorem of continuous analysis that is used in physical prediction, there exists a finite analog — stated in terms of bounded rational arithmetic with explicit error terms — that yields the same numerical predictions within any specified tolerance." This is a *metatheorem* about the relationship between the finite system and the continuous one it replaces. It's not trivial to prove in full generality, but versions of it are implicit in every applied mathematics and numerical methods textbook ever written. The practice of engineering *is* the evidence — every bridge that stands, every circuit that functions, every weather prediction that's roughly right is a confirmation that finite computation recovers what the continuous formalism promises. The paper's job is to make this explicit rather than leaving it as a gesture toward "very good approximation."

Third: Can a purely finite mathematics match infinity's power as a discovery engine?

This is the hardest of the three to answer, and the paper shouldn't pretend it has a clean response. The Weil conjectures example is the sharpest version of the problem: mathematicians reasoning about the Riemann zeta function over $\mathbb{C}$ noticed patterns that led them to conjecture analogous results for zeta functions over finite fields, and Deligne proved those finite-field results using machinery (étale cohomology) that was itself inspired by the infinite framework. The infinite detour produced finite knowledge. Can the paper account for this?

Partially. The paper can observe that analogy and heuristic reasoning don't require the objects of the analogy to exist. Physicists use the frictionless plane and the point mass as reasoning tools without claiming that either exists. Mathematicians could use $\mathbb{C}$ and the Riemann zeta function as *heuristic devices* — conceptual scaffolding that suggests conjectures about finite objects — without committing to their ontological reality. The scaffolding is useful even if it doesn't correspond to anything. This is a coherent position, and it's one that many working mathematicians probably already hold implicitly.

But there's a harder version of the objection that this doesn't fully address: is there something about the *structure* of infinite mathematics that makes it uniquely good scaffolding? If so, that structural effectiveness might itself be evidence that the infinite framework is tracking something real — not physical reality, perhaps, but some kind of structural reality that finite mathematics alone can't access as efficiently. The paper doesn't need to resolve this question definitively, but it should acknowledge the force of it and note that distinguishing between "infinity is ontologically real" and "infinity is an extraordinarily effective cognitive tool for finite minds reasoning about very large finite structures" is a genuinely open philosophical question. The paper's position is compatible with the latter. It's not compatible with the former, and it should be clear-eyed about the fact that some of the evidence (like the Weil conjectures case) sits uncomfortably between the two.

These are not fatal objections. They're the research program that the paper, if taken seriously, would need to generate.